Speed-up reliability assessment for multi-component systems: importance sampling adapted to piecewise deterministic Markovian processes

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ABSTRACT: The reliability of a complex industrial system can rarely be assessed analytically. As system failure is often a rare event, crude Monte-Carlo methods are prohibitively expensive from a computational point of view. Methods such as importance sampling or multilevel splitting algorithms can be used to reduce computation times. We propose an adaptation of the importance sampling method for a class of multi-component dynamical systems. We focus on systems, which failure corresponds to a physical variable of the system (temperature, pressure, water level) entering a critical region. Reliability assessment of such systems requires an accurate model of the trajectory of the physical variables. As the statuses of the components (on, off, or out-of-order) often determine the trajectory of the physical variable of the system, the model should incorporate alterations of components statuses. One difficulty is that these alterations are caused by both deterministic feedback mechanisms and random failures or repairs. To handle these hybrid alterations, we first propose a model of the trajectory of the physical variables of the system based on piecewise deterministic Markovian processes (PDMP) introduced by M.H.A. Davis. Then we show how to adapt the importance sampling method to our model. We present a strategy to apply importance sampling to PDMP. A simulation study, carried out with the python library PyCATSHOO developed by EDF R&D, compares our importance sampling method to the crude Monte-Carlo method.

1 INTRODUCTION

For both safety and quality issues, nuclear, hydraulic and other industries resort to probabilistic safety assessment to quantify the reliability of their systems. In recent years, dynamic reliability methods have been gaining interest as they allow to better capture the influence of time and to process the dynamics of complex systems. When treating with complex industrial systems, reliability analysis faces two main challenges : the first challenge is related to the modeling of the systems which has to encounter for dynamics and inherent behavior of the system, the second one concerns the quantification of reliability as only numerical studies are possible due to the complexity of the systems.

1.1 *A model based on a PDMP*

In many industrial systems, failure corresponds to a physical variable of the system (temperature, pressure, water level) entering a critical region. The physical variables usually can enter this region only if a sufficient number of components of the system are damaged. In order to estimate dynamic reliability we need a model of the trajectory of the physical variable and of the statuses of components. We will call this pair of vectors the state of the system. To address the challenge of modeling the trajectory of state, we fall in with the work of (Zhang et al. 2015), as we model the evolution of the state of a system by a piecewise deterministic Markovian process (PDMP). PDMPs were introduced by (Davis. 1984, Davis 1993), they are meant to represent a large class of Markovian processes that do not include diffusion, and as such they benefit from high modeling capacity. These processes can easily incorporate component aging, failure on demand, and delays before repairs.

1.2 *A rare event issue*

The second challenge, is that reliability of a complex industrial system can rarely be assessed analytically, so reliability analysis often relies on Monte-Carlo simulations techniques. But in the context of reliable systems, crude Monte-Carlo techniques perform poorly because system failure is a rare event. Indeed, when the reliability approaches zero, the number of simulations needed to get a reasonable precision on the relative error increases dramatically with

the Monte-Carlo method, and so does the computational time. To reduce this computational burden, one option is to reduce the number of simulations needed by using a variance reduction method. Amongst variance reduction techniques we may think at the multilevel splitting techniques and at importance sampling techniques. In this article we provide an adaptation of importance sampling to PDMP trajectories.

1.3 *Prerequisite for importance sampling*

The idea of importance sampling consists in simulating from a more fragile system, while eliminating the induced bias by the weighting simulation outputs by a likelihood ratio. To define such a likelihood ratio for PDMP trajectories it is necessary to dispose of a measure dominating both the law of the trajectories of our system and the law of the importance system (i.e. the weaker system used for simulations). Indeed PDMP are very degenerated processes, their law involve hybrid random variables which have continuous and discrete parts, it is important to insure we do have a dominating measure to define the likelihood ratio properly.
In our case the state of the system at time t is noted Z_t, and is given by both the random variables values, gathered in the vector X_t, and the statuses of all the components in the system, represented by a vector M_t, so that $Z_t = (X_t, M_t)$. Throughout the paper we call X_t the position of the system, and M_t the mode of the system. $\mathbf{Z}$ represents a trajectory of our PDMP up to a final observation time t_f. We consider all trajectories $\mathbf{Z}$ that are initiated in state z_o. If A is the critical region corresponding to system failure, then we note $\mathcal{A}$ the set of trajectories on $[0; t_f)$ that pass through A. We want to estimate $\mathbb{P}\big(\mathbf{Z} \in \mathcal{A} | Z_0 = z_o\big) = \mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{A}\big)$. Suppose ζ is a dominating measure for $\mathbb{P}_{z_o}\big(\mathbf{Z} \in .\big)$, and f is the density of the trajectory $\mathbf{Z}$ with respect to ζ, and g is the density of the trajectory of an importance process with respect to ζ. If ζ exists, and $\forall \mathbf{z} \in \mathcal{A},\ f(\mathbf{z}) \neq 0 \Rightarrow g(\mathbf{z}) \neq 0,$ then we can write :

$$\mathbb{P}\Big(\mathbf{Z} \in \mathcal{A}\Big) = \mathbb{E}_f\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big] = \int_{\mathcal{A}} f(\mathbf{z})\, d\zeta(\mathbf{z})$$

$$= \int_{\mathcal{A}} \frac{f(\mathbf{z})}{g(\mathbf{z})} g(\mathbf{z})\, d\zeta(\mathbf{z}) \qquad (1)$$

and $\mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{A}\big)$ can be estimated without bias by:

$$\hat{p}_{IS} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \mathbb{1}_{\mathcal{A}}(\mathbf{Z}'_i) \frac{f(\mathbf{Z}'_i)}{g(\mathbf{Z}'_i)} \qquad (2)$$

$$\text{where } \mathbf{Z}'_i \overset{iid}{\sim} g$$

Thus the use importance sampling on PDMP trajectories requires the following three conditions :

(C1) We dispose of a measure ζ on the trajectory space, and the trajectory $\mathbf{Z}$ of the system's state has density f with respect to ζ

(C2) We are able to simulate trajectories according to an importance process $\mathbf{Z}'$ which has density g with respect to ζ on $\mathcal{A}$.

(C3) $\forall \mathbf{z} \in \mathcal{A},\ f(\mathbf{z}) \neq 0 \Rightarrow g(\mathbf{z}) \neq 0$

When $\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z}) \frac{f(\mathbf{Z})}{g(\mathbf{Z})}\big] < \infty$ and the conditions above are verified, we have a central limit theorem on $\hat{p}_{IS}$. The estimator $\hat{p}_{IS}$ is unbiased and $\sqrt{N}(\hat{p}_{IS} - p)$ converges in law to a centered Gaussian with variance $\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z}) \frac{f(\mathbf{Z})}{g(\mathbf{Z})}\big] - p^2$. Theoretically the variance can be brought to zero if the importance density is $g(\mathbf{z}) = \frac{\mathbb{1}_{\mathcal{A}}(\mathbf{z}) f(\mathbf{z})}{p}$, but this can be done only if we already dispose of the value of p which we are trying to estimate. In practice one tries to approach this optimal density choosing $g(\mathbf{z}) \simeq \frac{\mathbb{1}_{\mathcal{A}}(\mathbf{z}) f(\mathbf{z})}{p}$ to reduce the variance.
Many authors have used importance sampling on particular cases of PDMP sometimes without noting it, see (Lewis and Böhm 1984) (Labeau 1996a) (Labeau 1996b) (Marseguerra and Zio 1996). Most of the models considered in literature rarely used all the possibilities offered by PDMPs, as they only considered constant failure and repair rates, not taking into account component aging and dependency of failure rate on physical conditions, or they avoid considering automatic control mechanisms which activate and deactivate components depending on the values of physical variables. In (Marseguerra and Zio 1996), and in the more recent work of (Ramakrishnan 2016) importance sampling is used on PDMP while taking into account automatic control mechanisms but they do not clearly identify the dominating measure. In this paper we show how to find a dominating measure ζ defining the likelihood ratio and we try to highlight the possible kinds of importance processes associated with ζ.

The rest of the paper is organized as follows: section 2 presents the generic PDMP model for multi-component systems and an example of two-components system, section 3 explains how to obtain the law of the trajectory of a PDMP and a dominating measure with the associated densities, section 4 investigates some possible kinds of important process, and finally section 5 presents a comparison between importance sampling and Monte-Carlo on the two-system presented in section 2.

2 A MODEL FOR MULTI-COMPONENT SYSTEM BASED ON PDMP

2.1 *Possible state of the system*

We consider a system with N_c components and d physical variables. Remember positions the vector $X \in \mathbb{R}^d$ which represents physical variables

of the system, and we call mode the vector $M = (M^1, M^2, ..., M^{N_c})$ gathering the statuses of each of the N_c components. The status of a component can be alternatively ON, or OFF or out-of-order, so that the set of modes $\mathbb{M}$ is $\mathbb{M} = \{On, Off, Failed\}^{N_c}$. As long as $\mathbb{M}$ stays countable, it is possible to consider more options for the statuses of components. For instance one could consider different regimes of activity instead of the simple status On, or different types of failure instead of the status $Failed$. Due to automatic control mechanisms, in a given mode m the physical variables are restricted to an open and connected set $\Omega_m \subset \mathbb{R}^d$. With $E_m = \{(x, m), x \in \Omega_m\}$, the state space is then:

$$E = \bigcup_{m \in \mathbb{M}} E_m = \bigcup_{m \in \mathbb{M}} \Big\{(x, m), x \in \Omega_m\Big\}$$

2.2 *Flow functions*

In a given mode m, i.e. a given combination of statuses of components, the evolution of position is determined by an ordinary differential equation. We note ϕ_x^m the solution of that equation initiated in x, if we consider a position X_t at time t, there exists a duration $T > 0$ such that $\forall s \in [0; T)$, $X_{t+s} = \phi_{X_t}^{M_t}(s)$. For an initial state $z \in E$ we also introduce a similar flow function Φ_z with values in E. So when we start from a state $Z_t = (X_t, M_t)$, then next states are locally given by Φ_{Z_t}:

$\exists T > 0, \forall s \in [0; T)$,

$$Z_{t+s} = \Phi_{Z_t}(s) = \big(\phi_{X_t}^M(s), M_t\big) = \big(X_{t+s}, M_t\big) \qquad (3)$$

This flow function allows us to describe the evolution of the state between two disruptions due to control mechanisms or failures and repairs.

2.3 *Jumps*

The trajectory of state can also evolve by jumping. When a jump is triggered, the current state moves to another one by changing its mode and/or its position. Jumps typically occur when there is a change in mode due to a control mechanism or a failure or a repair, or they can also occur when the physical variables are discontinuous. For instance, if we include the ages of components in the physical variables, a discontinuity of position occurs when a component is replaced and let in the same status.
We note $\overline{E}$ the closure of E, and is $\mathcal{E}$ the Borelian σ-algebra on E. Let say a jump occurs at time T, then the destination of the jump is determined according to a transition Kernel $K_{Z_T^-}$ where $Z_T^- \in \overline{E}$ is the departure state of the jump. If $Z_T^+ \in E$ be the arrival state, and if $\forall z^- \in \overline{E}$, ν_{z^-} is a measure on E, the kernel is defined by:

$$\forall B \in \mathcal{E}, \quad \mathbb{P}\left(Z_T^+ \in B | Z_T^- = z^-\right) = K_{z^-}(B)$$

$$\mathbb{P}\left(Z_T^+ \in B | Z_T^- = z^-\right) = \int_B K_{z^-}(z)\, d\nu_{z^-}(z) \qquad (4)$$

The kernel density must verify $K_z(z) = 0$ so we can not jump on the departure state.

2.4 *Time of the next jump*

Jumps at boundaries
For $m \in \mathbb{M}$, let $\partial\Omega_m$ be the frontier of Ω_m, we call boundary of set E_m the set $\partial E_m = \{(x, m), x \in \partial\Omega_m\}$. For $z = (x, m) \in E$, we define $t_z^* = \inf\{s > 0, \Phi_z(s) \in \partial E_m\}$ the time at which the flow would hit the boundary. We take the convention $t_z^* = +\infty$ if $\{s > 0, \Phi_z(s) \notin E_m\} = \emptyset$. Consider the system starts in state $z = (x, m)$. When the flow leads the position out of its restricted set Ω_m, an automatic jump is triggered. In other words, when the trajectory of the state of the system hits the boundary ∂E_m at time t_z^*, a jump is triggered.

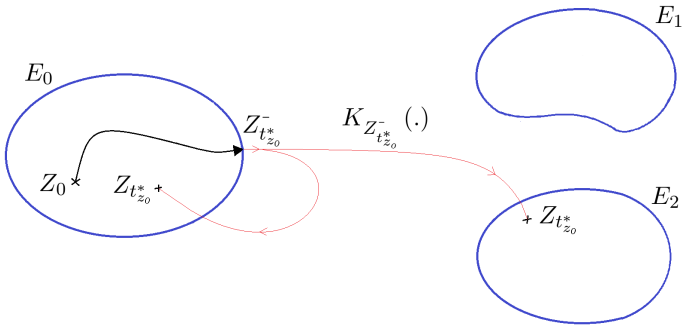

Figure 1: A jump at boundary,

Boundaries can be used to model automatic control mechanisms, or any automatic change in the status of a component. For instance in a dam, if the water level X reaches a given threshold X_{max} the evacuation valve automatically opens, to avoid overflow. If $M = closed,\ Opened,\ failed$ represent respectively the modes where the valve is closed, or opened or failed, this control system could be modeled by setting $\Omega_{Closed} = (0; X_{max})$ and $K_{(X_{max}, Closed)}(\{(X_{max}, Opened)\}) = 1$.
Boundaries also allow to include failure on demand, by incorporating a probability of jumping to a failed mode at the boundary. In our example this would be modeled by setting $K_{(X_{max}, Closed)}(\{(X_{max}, Opened)\}) = 1 - p$ and $K_{(X_{max}, closed)}(\{(X_{max}, Failed)\}) = p$, where $p \in [0; 1]$ is the probability of failure on demand on this boundary.

Spontaneous jumps
The trajectory can also jump to another state when a random failure or a repair occurs (see Figure 2).

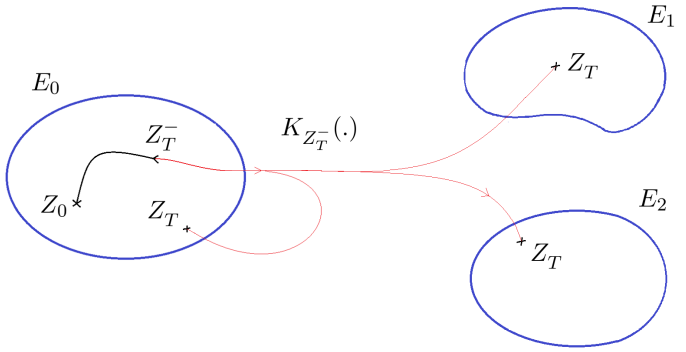

Figure 2: A spontaneous jump,

The time at which it happens if modeled through a state-related intensity function $\lambda : E \to \mathbb{R}_+$. For $z = (x, m) \in E$, $\lambda(z)$ represents the instantaneous risk of having a failure or a repair at state z. To simplify notations we note $\lambda_z(t) = \lambda(\Phi_z(t))$ and $\Lambda_z(t) = \int_0^t \lambda\big(\Phi_z(u)\big)du$. Starting the trajectory at a state z, we denote by T the time of the next jump and $\mathbb{P}_z(.)$ is the probability of an event knowing $Z_0 = z$. Thus we have:

$$\mathbb{P}_z(T \leq t) = \begin{cases} 1 - \exp\left[-\Lambda_z(t)\right] & \text{if } t < t_z^* \\ 1 & \text{if } t \geq t_z^* \end{cases} \tag{5}$$

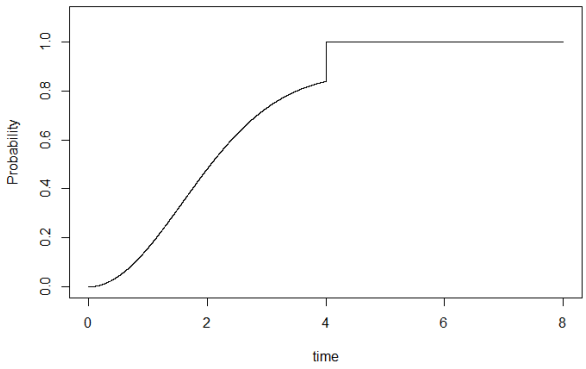

Figure 3: An example of the cdf of T, where $\lambda(z) = ax$ and $t_z^* = 4$

The law of T has a continuous and a discrete part (see Figure 3), so we introduce the measure

$$\mu_z(.) = leb(. \cap (0; t_z^*)) + \delta_{t_z^*}(.)$$

which will be useful to define the dominant measure ζ. We can also write the law of T under an integral form :

$$\mathbb{P}_z(T \leq t) =$$

$$\int_{(0;t]} \Big(\lambda_z(u)\Big)^{\mathbb{1}_{u \neq t_z^*}} \exp\Big[-\Lambda_z(u)\Big] d\mu_z(u) \tag{6}$$

Note that equations (5) or (6) give the time of the next jump, would it be a failure or a repair or an automatic activation or deactivation. If $T < t_z^*$ then the jump corresponds to a failure or a repair, but (5) or (6) do not tell which component has been repaired or failed. To specify it we use the transition kernel $K_{Z_t^-}$. Let T^j be the duration before the first failure or repair of component j if the physical variables had followed the flow ϕ_x^m indefinitely. Let $\lambda^j : E \to \mathbb{R}_+$ be its associated state-related intensity function, such that $\mathbb{P}_z(T^j \leq t) = 1 - \exp\left[\int_0^t \lambda^j\big(\Phi_z(u)\big)\right] du$. The function λ^j is either a failure rate or a repair rate depending on the initial value of m^j in z which gives the initial status of component j. Knowing the initial state, and therefore knowing the indefinite trajectory of positions given by ϕ_x^m, we make the assumption that the T^j's are independent. This assumption is true if the position gathers all the variables affecting failure or repair. If another variable y affect the failure or repair rate of several components the assumption is false. As $T = \min\{T^1, T^2, \ldots, T^{N_c}, t_z^*\}$, this conditional independence allows us to say that:

$$\forall z = (x, m) \in E, \quad \lambda(z) = \sum_{j=1}^{N_c} \lambda^j(z) \tag{7}$$

If $j(m \to m^+)$ is the index of the component which failed or was repaired when system switched from mode m to mode m^+ we have:

$$\forall z^+ = \big(x^+, m^+\big) \in E,$$

$$K_z\big(z^+\big) = \frac{\lambda^{j(m \to m^+)}(z)}{\sum_{j=1}^{N_c} \lambda^j(z)} q_z\big(x^+\big) \tag{8}$$

where $q_z(x^+)$ is the density of a transition kernel for positions. Typically if the physical variable are all continuous then $q_z(x^+) = \mathbb{1}_{x=x^+}$ and the reference measure of the transition kernel is defined by $\forall B \in \mathcal{E}, \quad \nu_z(B) = \sum_{w \in \mathbb{M} \setminus \{m\}} \delta_{(x,w)}(B)$.

2.5 *Generate a trajectory*

To generate a realization of the PDMP on can follow these steps:

- Initiate trajectory at state $Z_0 = z$
- Generate T the time of the next jump and follow the flow Φ until this time
- Generate $Z_T = z_T$ the arrival state of the jump
- Repeat starting with z_T until you pass observation time t_f

2.6 *Example*

As an example we can consider a room heated by two heaters. X_t represents the temperature of the room at time t. X_e is the exterior temperature. β_1 is the rate of

heat transition with exterior. β_2 is the heating power of each heater. The heaters are programmed to maintain the temperature within an interval $(X_{min}; X_{max})$ where $X_e < X_{min}$. Heaters can be on off or out-of-order : $\mathbb{M} = \{On, Off, Failed\}^2$. We consider that the two heaters are in passive redundancy: the second heater can be activated only if the first one is failed.

The differential equation giving the evolution of the position is

$$\frac{d\,X_t}{dt} = \beta_1(X_e - X_t) + \beta_2 \mathbb{1}_{M^1 \, or \, M^2 = On}$$

The failure rates of heaters depend on position:

$$\lambda^j(X, M) = 0.005\frac{X}{X_{max}} \quad \text{when } M^j = On$$

We consider exponential repair times:

$$\lambda^j(X, M) = 0.2 \quad \text{when } M^j = Failed$$

To handle the programming of the heaters when $M \in \{(Off, Off, Off), (Failed, Off), (Off, Failed)\}$ we set $\Omega_M = (-\infty; X_{max}), and otherwise \Omega_M = (X_{min}; X_{max})$. Due to continuity of temperature, the reference measure for the kernel is $\forall B \in \mathcal{E},\ \nu_{(x,m)}(B) = 1$ if $\exists z^+ = (x^+, m^+) \in B$ such that $x^+ = x$ and $m^+ \neq m$, and $\nu_{(x,m)}(B) = 0$ otherwise. The transition kernel satisfies equation (8) with $q_x(x^+) = \delta_x(x^+)$, and for z on boundaries $K_z\big(z^+\big) = 0$ or 1 according to the heaters programming.

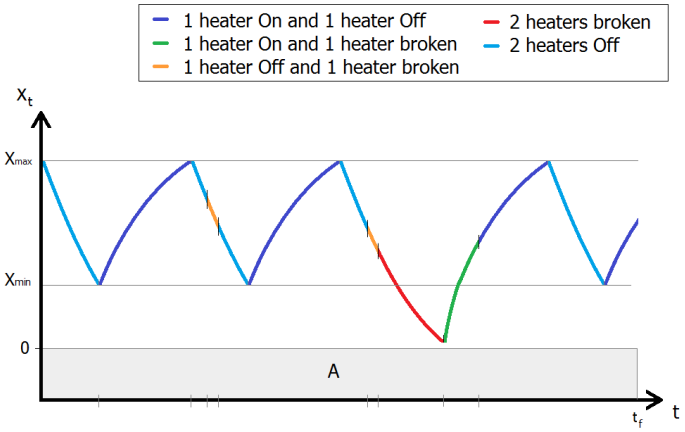

Figure 4: Scheme of a trajectory of the two-component system (mode is represented with colors)

3 IMPORTANCE SAMPLING ADAPTATION

We begin this section by introducing a few notations: For a trajectory $\mathbf{Z}$ on $[0; t_f)$, we note N the number of jumps before t_f, and S_k the time of the k-th jump with the convention $S_0 = 0$, and $\forall k < N,\ T_k = S_{k+1} - S_k$ is the duration between two jumps and $T_N = t_f - S_n$ is the remaining time until t_f after the last jump. One can easily verify the sequence of the $(Z_{S_k}, S_{k+1} - S_k)$ is a Markov chain, it is called the embedded Markov chain of the PDMP.

3.1 *The law of trajectories*

The main idea in building the law of the trajectory $\mathbf{Z}$ is to summarize the trajectory by the truncated embedded Markov chain of the process: the vector $\big(Z_{S_0}, T_0, Z_{S_1}, T_1, \ldots, Z_{S_N}, T_N\big)$. As the trajectory is piecewise deterministic we only need to keep the states of arrival of jumps and the durations between jumps to describe the randomness of the trajectory. Inversely if we have $\big(Z_{S_k}, T_k\big)_{k \leq N}$ and we know the flow function Φ we have enough information to reconstruct the trajectory using (3). Noting Θ the map that changes $\mathbf{Z}$ in $\big(Z_{S_k}, T_k\big)_{k \leq N}$, the law of $\mathbf{Z}$ can be defined as the image law of $\big(Z_{S_k}, T_k\big)_{k \leq N}$ through Θ. As (6) gives the law of T_k knowing Z_{S_k}, and as $Z^-_{S_{k+1}} = \Phi_{Z_{S_k}}(T_k)$, (4) gives the law of $Z_{S_{k+1}}$ knowing $\big(Z_{S_k}, T_k\big)$, using the Markov structure of sequence $\big(Z_{S_k}, T_k\big)_{k \leq N}$ we can get its law.
Defining the σ-algebra $\mathcal{S}$ on set of possible values of $\big(Z_{S_k}, T_k\big)_{k \leq N}$ as the σ-algebra generated by the sets in $\bigcup_{n \in \mathbb{N}^*} \mathcal{B}\Big(\Big\{\big(z_{s_k}, t_k\big)_{k \leq n} \in (E \times \mathbb{R}_+^*)^n, \sum_{i=0}^{n} t_i = t_f\Big\}\Big)$. Where $\mathcal{B}(.)$ indicates the Borelians of a set. We get that for $B \in \mathcal{S}$:

$$\mathbb{P}_{z_o}\Big(\mathbf{Z} \in \Theta^{-1}(B)\Big) =$$

$$\int_B \prod_{k=0}^{n} \Big(\lambda_{z_k}(t_k)\Big)^{\mathbb{1}_{t_k \neq t^*_{z_k}}} \exp\Big[-\Lambda_{z_k}(t_k)\Big] \prod_{k=1}^{n} K_{z_k^-}(z_k)$$

$$\times\, d\nu_{z_n^-}(z_n)\, d\mu_{t^*_{z_{n-1}}}(t_{n-1}) \ldots d\nu_{z_1^-}(z_1)\, d\mu_{t^*_{z_o}}(t_0) \quad (9)$$

Where $z_j^- = \Phi_{z_{j-1}}(t_{j-1})$. Note that with our construction this is a probability law on the space of trajectories that verify (3) only.

3.2 *The dominant measure and the density*

We define the measure ζ so that

$$\zeta(\Theta^{-1}(B)) =$$

$$\int_{(z_k, t_k)_{k \leq n} \in B} d\nu_{z_n^-}(z_n)\, d\mu_{t^*_{z_{n-1}}}(t_{n-1}) \ldots d\nu_{z_1^-}(z_1)\, d\mu_{t^*_{z_o}}(t_0) \quad (10)$$

If $\forall z \in \bar{E},\ \nu_z$ is finite and $t_f < \infty$, then ζ is a σ-finite measure, and by Radon-Nikodym theorem the density of a trajectory $\mathbf{z} = \Theta\big((z_0, t_0), \ldots, (z_n, t_n)\big)$ is

$$f(\mathbf{z}) = \prod_{k=0}^{n} \Big(\lambda_{z_k}(t_k)\Big)^{\mathbb{1}_{t_k \neq t^*_{z_k}}} \exp\Big[-\Lambda_{z_k}(t_k)\Big] \prod_{k=1}^{n} K_{z_k^-}(z_k) \quad (11)$$

4 POSSIBLE IMPORTANCE PROCESSES

4.1 *Conditions*

A possible importance process is a process which law is absolutely continuous with respect to ζ (condition C2) and which satisfies $\forall \mathbf{z} \in \mathcal{A},\ f(\mathbf{z}) \neq 0 \Rightarrow g(\mathbf{z}) \neq 0$ (condition C3).

4.2 *In practice*

The elements relative to the importance process are noted with a $'$. Condition C2 means the importance process must generate trajectories in E which satisfy equation (3). Also we need to be available to generate the importance process. To do so we will consider that the value of a point of the trajectory Z'_t only depends on past values even though it is restrictive. In other words, we keep generating trajectories by successively generating the arrival state of a jump (Z'_{S_k}) and the time until the next jump (T'_k), but now the law of Z'_{S_k} can depend on $\big(Z'_{S_i}, T'_i\big)_{i<k}$ and the law of T'_k can depend on $\big(Z'_{S_i}, T'_i\big)_{i<k}$ and Z'_{S_k}. Note that in order to insure condition C2 the law of T'_k still has to be dominated by $\mu_{Z'_{S_k}}$, and the law of $Z'_{S_{k+1}}$ still has to be dominated by $\nu_{Z'^{-}_{S_k}}$. If $\underline{Z}'_k = \big((Z'_{S_i}, T'_i)_{i<k}, Z'_{S_k}\big)$ and if $\lambda'_{\underline{z}_k}$ is the intensity function associated to T'_k then:

$$\mathbb{P}(T'_k \leq t | \underline{Z}'_k = \underline{z}_k) =$$

$$\int_{(0;t]} \Big(\lambda'_{\underline{z}_k}(u)\Big)^{\mathbb{1}_{u \neq t^*_{\underline{z}_k}}} \exp\Big[-\Lambda'_{\underline{z}_k}(u)\Big] d\mu_{z_k}(u) \qquad (12)$$

Another possibility is that the intensity function of T'_k does not have to be of the form $\lambda' \circ \phi_{z_k}$ where λ' would be a positive function on E. This means that at the time t the intensity function only depends on the arrival state of the last jump and on previous pairs (Z'_{S_i}, T'_i), and not on the state Z'_t as it is the case in a PDMP.

Concerning Condition C3 things are more delicate, indeed with big systems the set $\mathcal{A}$ can be very hard to apprehend, so we will only provide a sufficient condition to satisfy C3. To verify condition C3 a sufficient condition is to impose for any $z_k \in E$ and $z^- \in \overline{E}$:

$$\lambda(\Phi_{z_k}(t)) > 0 \ \Rightarrow \lambda'_{\underline{z}_k}(t) > 0$$

$$K'_{z^-}(z^+) > 0 \ \Rightarrow K'_{z^-}(z^+) > 0$$

4.3 *Parametric bias*

To find an importance process that gives a good variance reduction, we usually consider a parametric family of importance densities. We can then optimize the variance reduction by using the cross-entropy method presented in (Boer et al. 2005) to select the parameters of the importance density. However, it is possible that minimization routines used in the cross entropy method converge to local optima. Therefore, to avoid this phenomenon one should run several times the cross entropy method with different initial values for the vector of parameters. Note that the parametrization must be chosen carefully: indeed the family of importance densities must contain densities that are close to the zero-variance density $g(\mathbf{z}) = \frac{\mathbb{1}_{\mathcal{A}}(\mathbf{z}) f(\mathbf{z})}{p}$ to obtain a good variance reduction, otherwise we could even obtain a variance increase. Ideally we want to specify parametric densities that increases the likelihood of *all* elements in $\mathcal{A}$. In practice, we increase the probability of $\mathcal{A}$, while trying to simulate the elements of $\mathcal{A}$ in proportion to their natural probabilities.

4.4 *State set expansion trick*

Before considering biasing strategies, it can be convenient to consider the PDMP of the system as a PDMP $\tilde{\mathbf{Z}}$ in larger state space $\tilde{E}$. We note with a $\sim$ the element relative to the extended PDMP $\tilde{\mathbf{Z}}$. By 'larger' state space, we mean that the position and mode of the states in $\tilde{E}$ have additional coordinates that we note $\tilde{x}$ and $\tilde{m}$. We dispose of a projection π mapping elements of $\tilde{E}$ on those of E.

$$\begin{array}{rccc} \pi\ : & \tilde{E} & \longrightarrow & E \\ & \big((x,\tilde{x}),(m,\tilde{m})\big) & \longrightarrow & \big(x,m\big) \end{array}$$

Each trajectory of the process $\tilde{\mathbf{Z}}$ can be projected to get a trajectory of the process $\mathbf{Z}$. The process $\tilde{\mathbf{Z}}$ is a PDMP on the set $\tilde{E}$ equipped with a σ-algebra $\tilde{\mathcal{E}}$ such that we can identify $\pi(\tilde{\mathbf{Z}})$ and $\mathbf{Z}$, i.e. if $\mathcal{T}$ is a set of trajectories on E with $\tilde{\mathcal{T}} = \pi^{-1}(\mathcal{T})$ then $\mathbb{P}_{\tilde{z}_o}\big(\tilde{\mathbf{Z}} \in \tilde{\mathcal{T}}\big) = \mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{T}\big)$ where $z_o = \pi(\tilde{z}_o)$ .To be an extension of the process $\mathbf{Z}$, the process $\tilde{\mathbf{Z}}$ must verify the following conditions:

- $\forall \tilde{z} \in \tilde{E},\ \pi\big(\tilde{\Phi}(\tilde{z},t)\big) = \Phi(\pi(\tilde{z}),t)$
- $\forall \tilde{z} \in \tilde{E},\ \tilde{\lambda}_{\tilde{z}}(t) = \lambda_{\pi(\tilde{z})}(t)$
- $\forall \tilde{z} \in \tilde{\overline{E}},\ \forall \tilde{B} \in \tilde{\mathcal{E}}$ such that $\pi(\tilde{B}) \in \mathcal{E}$, $\tilde{K}_{\tilde{z}}(\tilde{B}) = K_{\pi(\tilde{z})}(\pi(\tilde{B}))$

Once we have built such a process we can generate it to get an estimation of $\mathbb{P}_{\tilde{z}_o}\big(\tilde{\mathbf{Z}} \in \pi^{-1}(\mathcal{A})\big) = \mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{A}\big)$.

The benefit of that expansion of the state space is that when we do importance sampling on the extended process, the possible importance processes can now depend on the additional coordinates in position and mode, yielding more possible biasing

strategies. For instance, if $\tau_A = \inf\{t > 0, Z_t \in A\}$, consider the extended process $\tilde{Z}_t = \big(X_t\,, (m_t, \mathbb{1}_{\tau_A<t})\big)$ would allow to change (or stop) bias after the event, consider $\tilde{Z}_t = \big((X_t, t), m_t\big)$ would allow to increase or decrease biasing around some time, consider $\tilde{Z}_t = \big(X_t, (m_t, n_t)\big)$ where n_t is the number of jumps would allow to change the strength of bias after each jump.

5 SIMULATION STUDY

5.1 *The original system*

To show it is possible to find efficient importance processes we conduct a simulation study on the two-component system presented in section 2.6. The constants of the system are set to $X_{min} = 0.5,\ X_{max} = 5.5,\ X_e = -1.5,\ \beta_1 = 0.1,\ \beta_2 = 5$, and the critical region is all negative temperatures $A = (-\infty; 0]$.

5.2 *Choice of the parametric family*

To increase the probability of $\mathcal{A}$ we simply decrease the repair rate by replacing it by a smaller parameter α_3. Then to sample the elements of $\mathcal{A}$ in proportion to their natural probabilities, we start by noticing two things: For a trajectory to be in $\mathcal{A}$, it must cross the boundary in X_{min} , so the system must experience a failure of each heater before temperature reaches X_{min}. Second a trajectory with a heater that stays broken for to long is less likely in the original system because the likelihood in (11) can be factorized by $\exp(-0.2\,t_r)$ where t_r is the duration during which the heater stays broken. We deduce from these observations that the most likely trajectories in $\mathcal{A}$ are among those where the two last failure before crossing the boundary occur close from to the boundary and as close as possible in time so that the terms $\exp(-0.2\,t_r)$ are maximized for each heater. To simulate the trajectories of $\mathcal{A}$ according to their natural likelihood, we increase the occurrence of a first failure when position is close to X_{min} by setting the first failure rate to $\alpha_1 \exp[X_{min} - x]$ and to get a second failure quickly we set the second failure rate to $(\alpha_1{+}\alpha_2) \exp[X_{min} - x]$. Also for in $\mathcal{A}$, once temperature reaches 0 we should stop biasing to get closer to the natural likelihood so we consider the extended PDMP $\tilde{Z}_t = \big(X_t\,, (m_t, \mathbb{1}_{\tau_A<t})\big)$ to avoid biasing after the event occurred in the importance processes. Finally we get the following parametric family of importance processes:

For $\tilde{z} = \big(x, (m, 0)\big)$:

$$\tilde{\lambda}'(\tilde{z}) = \begin{cases} \alpha_1 \exp[X_{min}\text{-}x] & \text{if 0 heater is failed} \\ (\alpha_1{+}\alpha_2) \exp[X_{min}\text{-}x] & \text{if 1 heater is failed} \\ \alpha_3 & \text{if 2 heaters are failed} \end{cases}$$

For $z = \big(x, (m, 1)\big)$:

$$\tilde{\lambda}'(\tilde{z}) = \tilde{\lambda}(\tilde{z}) \qquad \text{(no bias)}$$

The kernel still verifies equation (8) and is unchanged on boundaries.

We do not change the kernel on boundaries because its support is originally a singleton: a modification of the jump kernel in this context would enlarge its support and this would lead to simulate trajectories that are impossible for our system and so that are not in $\mathcal{A}$, which would result in a loss of computational time.

5.3 *Results*

The values of the parameters selected by the cross-entropy method are $\alpha_1 \simeq 0.054$, $\alpha_2 \simeq 0.354$, $\alpha_3 \simeq 0.0028$. A comparison between Monte-Carlo and the associated importance sampling estimate is presented in table 1, showing we gain a factor 6.4 in terms of standard deviation. Simulations have been carried out using the python library PyCATSHOO currently developed by EDF R&D. This library allows to model and simulate dynamic hybrid systems using distributed stochastic hybrid automata (Chraibi 2013), and so can simulate PDMP.

	$\hat{p}$	$\hat{\sigma}$	IC$\times 10^4$
MC	2.7×10^{-4}	5.3×10^{-5}	$[1.68; 3.71]$
IS	3.1×10^{-4}	8.2×10^{-6}	$[2.95; 3.27]$

Table 1: Results for 100000 simulations, exact reliability is 3.17×10^{-4}

6 CONCLUSION

We have presented a model for multicomponent systems based on PDMPs. In order to speed-up reliability assessment on such systems, we have adapted importance sampling to trajectories of PDMP. We have given a dominant measure for PDMP trajectories, allowing to properly define the likelihood ratio needed to apply the importance sampling method. The possible kinds of importance processes were discussed, and we propose a state expansion method which increases possibilities of biasing strategies. Tested on a two-component system our importance sampling method has shown good performance, dividing the standard deviation by a factor 6.4.

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